# Testing and Interval Estimation of Hazard Rate Change with Staggered Entry


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## Abstract

In part I of the paper we develop a likelihood-ratio based procedure for testing for a change in the hazard rate of the exponential lifetime distribution when the test subjects enter at random times and the Type I censoring occurs. First we show that the profile log-likelihood ratio process converges weakly to a non-stationary Gaussian process. By suitable transformation we show that the limiting process becomes the Ornstein-Uhlenbeck process and derive the critical values of the test by considering the tail distribution of the supremum of the Ornstein–Uhlenbeck process. We consider the power of the test by Monte Carlo simulation.

In part II, under the same setting as in part I we consider the interval estimation for the hazard rate change using the local likelihood ratio approach. We establish the weak convergence of the approximate local likelihood ratio process and obtain the closed form for the approximate distribution of the supremum of the limiting process which does not depend on the change point. By inverting the log likelihood function, we construct a confidence interval for the change point. Using Monte Carlo simulation we compare the empirical coverge probability of the confidence interval with nominal confidence level and discuss the effect of smoothing of the log-likelihood function on the coverage probability and length of the interval.


## 1 Testing for a change in the hazard rate with staggered entry

### 1.1 Model setup

Suppose that the patients enter the treatment at times $0 < \tau_1 < \tau_2 < \cdots,$ following a Poisson process with rate $\gamma$. Let $N$ be the total number of patients entered in the time interval $[0, T]$. For $i = 1, \ldots, N$, let $Y_i$ be the survival time of the $i$th patient; and suppose that the distribution of the $Y_i$ is of the form

$$f_Y(y) = \begin{cases} \lambda_1 e^{-\lambda_1 y} & \text{if } y < \nu; \\ \lambda_2 e^{-\lambda_1 \nu - \lambda_2 (y-\nu)} & \text{if } y \geq \nu, \end{cases}$$

where $0 < \lambda_1,\ \lambda_2,\ \nu$ are unknown parameters. That is, the failure rate may change at an unknown time $\nu$.

The model is irregular in that $\nu$ only has meaning if $\lambda_1 \neq \lambda_2$. Moreover, in this setup, testing the presence of change point $\nu$ is translated to testing the hypothesis $H_0 : \lambda_1 = \lambda_2$ vs. $H_1 : \lambda_1 \neq \lambda_2$. In the first part of the paper, we develop a test based on the observations

$$X_i = \min(Y_i, T - \tau_i)$$

and

$$\delta_i = \mathbf{1}\{Y_i \leq T - \tau_i\}$$

where $\mathbf{1}\{A\}$ denotes the indicator of $A$. Let $\ell\,[\lambda_1, \lambda_2, \nu]$ denote the log-likelihood function. Then the profile log-likelihood ratio statistic $\Lambda$ for testing $H_0$ versus $H_1$ for a fixed change point $\nu$ is

$$\Lambda(\nu) = \ell(\hat{\lambda}_1(\nu), \hat{\lambda}_2(\nu), \nu) - \ell(\hat{\lambda}, \hat{\lambda})$$

where $\hat{\lambda}_1(\nu)$ and $\hat{\lambda}_2(\nu)$ are the maximum likelihood estimators (MLE) of $\lambda_1$ and $\lambda_2$, respectively, and $\hat{\lambda}$ is the MLE of the common hazard rate $\lambda$ under $H_0$.

The test depends on two design parameters $0 < a < b < 1$ and rejects $H_0$ if the supremum of $\Lambda(\nu)$ where $aT \leq \nu \leq bT$ is larger than a certain critical value. Inclusion of $a$ and $b$ here is necessary because a change point at $\nu = 0$ or $\nu = T$ is indistinguishable from $H_0$ within the context of the model. Given so, the test only looks for changes that occur between $aT$ and $bT$.

### 1.2 Other studies

A log-likelihood based test for change point in the hazard rate is not new. In a different formulation, Matthews and Farewell (1982) considered the testing problem using uncensored observations. Based on Monte Carlo simulation, they suggested that moderate amount of Type I censoring has little impact on the null distribution of the test statistic.

Worsely (1988) obtained the exact null distribution of a restricted version of the likelihood ratio test statistic. He showed that the null distribution remains unchanged under Type II censoring. He also showed that the distribution heavily depends on the interval to which the change point is assumed to belong.

Loader (1991) considered the problem and concluded, using a heuristic argument, that the effect of random censoring on the significance level of the test to be relatively minor.

Matthews and Farewell (1985) studied the testing problem using the normalized score statistic and showed that the score process converges weakly to an Ornstein–Uhlenbeck process when the hazard rate is known and to a Brownian bridge when the hazard rate is unknown.

Our approach differs from these early studies in that the model allows random entry time (staggered entry) and the Type I censoring is explicitly built into the model.

### 1.3 Main result

It can be shown that twice the log-likelihood is equal to the square of a normalized process $Z_n(\nu)$, say, plus a remainder term that vanishes as $n$ tends to infinity. Furthermore, we can prove that under $H_0$, for any $0 < a < b < 1$, $Z_n(\nu)$ defined on $[a, b]$ converges in distribution to a Gaussian process $Z(\nu)$ on the same interval. The limiting process $Z(\nu)$ has zero mean and unit variance and the covariance function

$$\rho(\nu_1, \nu_2) = \sqrt{\frac{g_1(\nu_1 \wedge \nu_2) \cdot g_2(\nu_1 \vee \nu_2)}{g_1(\nu_1 \vee \nu_2) \cdot g_2(\nu_1 \wedge \nu_2)}}$$

for certain functions $g_1(\nu)$ and $g_2(\nu)$.

In order to obtain critical values for the test, we need to compute the tail probability of the supremum of the absolute value of $Z(\nu)$. In general, finding the supremum distribution of a nonstationary Gaussian process is a difficult problem and it usually requires extensive Monte Carlo simulation to determine the critical values. Fortunately, in this case we can find a suitable transformation that transforms the nonstationary Gaussian process to an Ornstein–Uhlenbeck process whose properties have been extensively studied in the literature. Approximate tail probability of $\sup_{a \leq \nu \leq b} |Z(\nu)|$ can be derived by the transformation and application of Theorem 12.2.9 in Leadbetter, Lindgren and Rootzén (1983).

### 1.4 Simulation

Monte Carlo simulation study shows that, not surprisingly, the sup distribution is far more affected by the length of the considered subinterval $[a, b]$ and much less so by the actual location of the interval. The power functions show U-shaped patterns, typical for two-sided tests. Interestingly, the power function are not symmetric. In general the test has greater power when $\lambda_1 < \lambda_2$ then when $\lambda_1 > \lambda_2$. This interesting asymmetry can be explained by the Hellinger distance. Since we would expect any properly constructed test to correctly reject the null hypothesis when the true underlying distribution is far from the null distribution, greater Hellinger distance imply higher power and in fact a routine calculation shows that the distance function is greater when $\lambda_1 < \lambda_2$ then when $\lambda_1 > \lambda_2$ even though the parameters may have the same absolute difference. Power function tends to decrease as the true change point is located on the right side of the considered interval.

## 2 Interval estimation for change point in the hazard rate

### 2.1 Model setup and background

In the same setup as above, we consider the problem of constructing a confidence interval for the change point. Two main approaches for the interval estimation of change point are based on: (1) limiting distribution of the MLE as Yao (1986) derived under some constraint, and (2) inversion of the log-likelihood process as in the works by Siegmund (1988) and Loader (1991).

By the jagged nature of the log-likelihood function, the inversion produces a confidence set consisting of disjoint intervals and Siegmund advocated the use of a single connected interval encompassing all the disjoint intervals and we follow his suggestion regarding confidence interval construction.

Loader applied the inversion technique to a change point confidence interval construction under exponential random censoring in a non-staggered entry scenario. Our work is different from their results in that the random entry time is allowed as well as Type I censoring is built into the model. Asymptotic theory based on local likelihood ratio process near the change point is also new.

### 2.2 Main result

The local likelihood ratio process $Z_\gamma(u)$ can be approximated by its two-term Taylor expansion $Z_\gamma^*(u)$ and we can prove that as $\gamma \to \infty$, $Z_\gamma^*(u)$ weakly converges to a process $Y(u)$, say, in the Skorohod space $D[a, b]$ for any $-\infty < a < b < \infty$. Furthermore, we can approximate the distribution of the supremum of $Y(u)$ by a function $F^*(x)$. That is, we can write

$$P\left\{\sup_u Y(u) \le x\right\} \approx F^*(x) = (1 - e^{-x})(1 - e^{-x}\kappa(\lambda_1/\lambda_2))$$

where

$$\kappa(x) = \left(\frac{1 - x + x\log x}{x - 1 - \log x}\right)^{-\mathrm{sgn}(\log x)}.$$

Notably, this approximating function is independent of the change point. For a given $\alpha > 0$, we can compute $c = c(\alpha)$ from the equation

$$F^*(c) = 1 - \alpha$$

and for this particular value of $c$, an approximate $100(1-\alpha)\%$ confidence interval for $\nu$ can be obtained by inverting the log-likelihood function.

### 2.3 Simulation

Monte Carlo simulation study shows that the confidence intervals constructed in this way tends to cover the true change point more often than the nominal coverage probability. To alleviate this excessive coverage probability of the confidence interval, we consider the nonparametric smoothing of the log-likelihood function prior to the inversion. We note that the usual criterion for smoothing such as cross validation does not work because the construction of change point depends on only a very small portion of the likelihood function and the data driven smoothing tends to completely wipe out the are, defeating the very purpose of smoothing. Evidences from simulation indicate that a small amount of smoothing is preferred because it substantially reduces the excessive coverage probability and shorten the average length of the confidence interval.

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